Fractional-order Integral State Space Modeling and Quasi State Analysis via Block Operational Matrix Scheme

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Abstract: In this paper, the fractional-order integral state space realization of a class of any proper fractional transfer function is presented. The operational matrix is generalized into the block form for fitting to carry out the quasi state analysis of the fractional-order integral state space model. Finally, some examples are exhibited to prove the validity of the proposed developments.

Key Words: Fractional-order System, Fractional-order Integral State Space Model, Chebyshev Polynomial, Block Operational Matrix

1 INTRODUCTION

Recently fractional-order systems have been revealed as a class of important dynamic systems [1–5]. The special dynamics they described are the long memory transient, heredity or non-locality [2], which have been found in electrochemistry [3], viscoelasticity [2], biological systems [4, 5], economics [1], etc. They are generally known to be infinite dimension and are hard for analysis. Two kinds of models are the fractional transfer function and the fractional-order differential equations, which are widely used [1]. However the fractional-order state space model is preferable for modeling, simulation and feedback control.

It is important to realize two difficulties in the fractional transfer function: (i) the higher order fractional transfer function is hard to be realized by both analog systems and numerical simulations; (ii) it do not keep track of what is going on internally in fractional-order systems. Therefore the realization of fractional-order state space models is proposed to resolve those drawbacks. As we know, the first application is the control of viscoelastically damped structures [6]. It is a special fractional-order state space realization for a commensurate fractional transfer function. An important progress was the pseudo-state space model [7], which is available for a specified fractional transfer function. A generalized fractional-order system was realized by the state space model in [8] by judging the transfer function of two canonical state space forms. To implement or simulate fractional-order systems, two approximation methods are proposed using CRONE approximation typically the fractional-order state space model and the fractional-order integral state space model [9, 10]. Another fractional-order state space realization is frequency distribute model, which is based on diffusive representation [11]. Besides, block pulse basis was introduced to realize fractional-order state space model in an algebraic way [12]. Although some modeling methods have been developed, the factional-order state space realization is much less counted. To extend and furnish the result in [7, 8], a generalized fractional-order state space realization for any proper fractional transfer function is developed. Correspondingly, a general fractional-order integral state space realization is proposed by including the initial conditions.

In our contributions, a class of any arbitrary high order fractional transfer functions can be realized by fractional-order integral state space models with several low fractional-order subsystems. They are easier to be handled in numerical or analog simulations than the fractional-order state space model [13]. Meanwhile, it provides a new approach to investigate the internal behavior of fractional-order systems. Block operational matrix of Riemann-Liouville integral with vector order is proposed to carry out the quasi state analysis of the fractional-order integral state space model. The main advantage is that the vector fractional-order differential equations can be approximated into a system of linear algebraic equations. Quasi state analysis is to determine the states with known fractional-order systems parameters and the input.

The analysis of quasi states in fractional-order state space models is called quasi state analysis as asserted in the title, a detail you can refer to [16]. Actually the stability analysis of fractional-order systems can be treated by analyzing the pseudo states [7]. Therefore, such pseudo states have some properties like the real states of fractional-order systems without being that, they are called quasi states in this paper.

The paper is organized as follows. Section 2 presents the fractional-order integral state space model. Block operational matrix of Riemann-Liouville integral with vector order is introduced in section 3 and the quasi state analysis is given. The numerical experiments are given in section 4. The final section is a conclusion.

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1. FRACTIOANL-ORDER INTEGRAL STATE MODEL

2.1 Fractional-order State Space Model

The conventional definitions of fractional calculus are used in this paper [1 – 3]. Riemann-Liouville integral and Caputo derivative of a function $f(t)$ are defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, n-1 < \alpha \le n$$

where $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$, α is fractional order.

Definition 1. A linear time invariant fractional-order system can be represented by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=1}^{F} b_i s^{t_i} + b_0}{\sum_{i=1}^{E} a_i s^{m_i} + a_0} \qquad (1)$$

where the fractional orders are $m_i, t_j \in \mathbb{R}^+, E, F \in \mathbb{N}^+$ and $m_1 < m_2 < \cdots < m_E, t_1 < t_2 < \cdots < t_F, t_F < m_E$. Without loss of generality, set $a_E = 1$. And denote $\Pi_1 = \{m_1, m_2, \cdots, m_E\}, \Pi_2 = \{t_1, t_2, \cdots, t_F\}$, Π_2 can be divided into two sets.

$$\Upsilon_1 = \{t_{r_i} \in \Pi_2, i = 1, \cdots, J \mid t_{r_1} < \cdots < t_{r_J}\}$$

$$\Upsilon_2 = \{t_{r_i} \in \Pi_2, i = 1, \cdots, K \mid t_{r_1} < \cdots < t_{r_K}\}$$

where $\Upsilon_1 \subset \Pi_1, \Upsilon_2 \cap \Pi_1 = \varnothing, K + J = F$.

Note $m_0 = 0$, the internal fractional orders are m_i, t_j, and the internal fractional orders are defined in two cases.

(i) $\Upsilon_2 = \varnothing$, the internal fractional orders are $n_i = m_i - m_{i-1}, i = 1, \cdots, E$.

(ii) $\Upsilon_2 \ne \varnothing$, the first E internal fractional orders are the case (i), the extra internal fractional orders are $n_i = m_i - m_{i-1}$ and

$$n_{E+i} = m_j - t_{r_i}, i = 1, \cdots, K,$$

$$m_{j-1} < t_{r_i} < m_j, \exists j \in \{1, \cdots, E\}$$

Let $A(s) = \sum_{i=1}^{E} a_i s^{m_i} + a_0, B(s) = \sum_{i=1}^{F} b_i s^{t_i} + b_0$

$$X(s) = \frac{U(s)}{A(s)}, Y(s) = B(s) X(s)$$

Theorem 1. Consider the general case (ii), the internal fractional orders are $n_i, i = 1, \cdots, E+K$. The fractional-order state space model can be represented by

$$\begin{cases} D^n Z(t) = AZ(t) + Bu(t) \\ y(t) = CZ(t) \end{cases} \qquad (2)$$

where the quasi state vector is $Z(t) = [z_1, \cdots, z_{E+K}]^T$,the fractional vector derivate is D^n with a fractional vector order $n = [n_1, \cdots, n_{E+K}]^T$. $A = \begin{pmatrix} A' & 0 \\ \Delta & 0 \end{pmatrix}$, the nonzero element of B is 1, Δ is determined by the internal fractional orders of additional quasi states and A' is a controllable canonical form with $A(s)$.

Proof. Choose the quasi states $z_1(t) = x(t), z_i(t) = D^{n_{i-1}} z_{i-1}(t), i = 1, \cdots, E$. We have

$$D^{n_i} z_i(t) = \cdots = D^{n_i + \cdots + n_1} z_1(t) = D^{m_i} x(t) = z_{i+1}(t)$$

Therefore, $X(s)$ can be expanded by

$$D^{n_E} z_E(t) = -a_{E-1} z_E(t) - \cdots - a_0 z_1(t) + u(t)$$

So far, the matrices A' and B can be obtained.

Consider $\Upsilon_2 \ne \varnothing$, the output $Y(s) = B(s) X(s)$ can be written by

$$y(t) = \sum_{i=1, t_{r_i} \in \Upsilon_1}^{J} b_{t_{r_i}} D^{m_{t_{r_i}}} x(t) + \sum_{i=1, t_{r_i} \in \Upsilon_2}^{K} b_{t_{r_i}} D^{m_{t_{r_i}}} x(t) + b_0 x(t)$$

Note $\Upsilon_1 \subset \Pi_1$, the first sum can be represented by above quasi states. However the second sum need the additional quasi states $z_{E+i}(t) = D^{t_{r_i}} x(t), t_{r_i} \in \Pi_2, i = 1, \cdots, K$. Thus we have

$$D^{n_{E+i}} z_{E+i}(t) = D^{m_j} x(t) = z_{j+1}(t) \qquad (3)$$

Now, the output can be written as

$$y(t) = \sum_{i=1, t_{r_i} \in \Upsilon_1}^{J} b_{t_{r_i}} z_{t_{r_i}}(t) + \sum_{i=1, t_{r_i} \in \Upsilon_2}^{K} b_{t_{r_i}} z_{t_{r_i}}(t) + b_0 z_1(t)$$

where two sums are represented by the quasi states and the additional ones respectively.

From (3), Δ can be obtained from the positions in Π_1 of additional quasi states. Thus the output matrix C can be determined accordingly.

So far, $E+K$ quasi states are chosen and all parameters in (2) are obtained.

A simple fractional-order state space realization of the case (i) is the following corollary.

Corollary 1. Consider the case (i), the internal fractional orders are $n_i, i = 1, \cdots, E$. The fractional-order state space model have the form (2), but the parameters have these forms.

$$Z(t) = [z_1, \cdots, z_E]^T, n = [n_1, \cdots, n_E]^T$$

$$A = A', B = [0, \cdots 0, 1]^T, C = [b_0, \cdots, b_F, 0, \cdots, 0]^T$$

Proof. It is obvious that the case (i) is special case of case (ii). And this result is obtained accordingly.

For simplicity, the dimension of fractional-order state space is assumed to be E in the subsections.

2.2 Fractional-order Integral State Space Model

To keep the initial conditions in the fractional-order integral state space model, the initial conditions of each quasi state are denoted by

$$\overline{z}_i(0)=\left[z_i(0),z_i'(0),\cdots,z_i^{(p_i-1)}(0)\right],p_i=\lceil n_i\rceil$$

Define the initial vectors $\rho_i=\left[1,t,\cdots,\frac{t^{p_i-1}}{(p_i-1)!}\right]^T$, the quasi states are equivalent to

$$z_i(t)=I^{n_i}w_i(t)+\overline{z}_i(0)\rho_i$$

In this paper, only the initial conditions in Caputo sense are considered, for Riemann-Liouville kind you can find in [9]. A general fractional-order integral state space models with dimension E is established.

Definition 2. The fractional-order integral state space model can be expressed by

$$\begin{cases}W(t)=AI^nW(t)+Bu(t)+AW(0)\\ y(t)=CI^nW(t)+CW(0)\end{cases}\tag{4}$$

where $W(0)=\left[\overline{z}_1(0)\rho_1,\cdots,\overline{z}_E(0)\rho_E\right]^T$ is the implicit initial condition. The quasi state vector is $Z(t)=I^nW(t)+W(0)$.

2. QUSI STATE ANALYSIS

3.1 Block Operational Matrix of Riemann-Liouville Integral

The Chebyshev polynomials defined on $[-1,1]$ have this recurrence formula [14].

$$T_{i+1}(x)=2xT_i(x)-T_{i-1}(x),T_0(x)=1,T_1(x)=x$$

Definition 3. By use of change $x=2t/L-1,t\in[0,L]$, the shifted Chebyshev polynomials are defined.

$$p_{i+1}(t)=2(2t/L-1)p_i(t)-p_{i-1}(t)$$

and $p_0(t)=1,p_1(t)=2t/L-1$.

Any square-integrable function $f(t)$ on $[0,L]$ can be expanded in terms of shifted Chebyshev polynomials with a truncation length m, $f(t)\simeq C^T\phi(t)$, where $C^T=[c_0,\cdots,c_m]$ is the shifted Chebyshev weight vector and $\phi(t)=[p_0(t),\cdots,p_m(t)]^T$ is the shifted Chebyshev vector. The shifted Chebyshev weights are given by

$$c_i=\frac{2}{\pi d_i}\int_0^L\frac{p_i(t)f(t)}{\sqrt{Lt-t^2}}dt,d_0=2,d_i=1,i=1,2,\cdots$$

Lemma 1. The operational matrix of Riemann-Liouville integral with an order α takes this form I_α.

$$I_\alpha=\begin{pmatrix}\sum_{k=0}^{0}\theta_{0,0,k} & \cdots & \sum_{k=0}^{0}\theta_{0,m,k}\\ \vdots & \ddots & \vdots\\ \sum_{k=0}^{m}\theta_{m,0,k} & \cdots & \sum_{k=0}^{m}\theta_{m,m,k}\end{pmatrix}$$

where $\theta_{i,j,k}$ can be expressed by

$$\frac{(-1)^{i-k}2iL^\alpha(i+k-1)!\Gamma(k+\alpha+0.5)}{d_j\Gamma(k+0.5)(i-k)!\Gamma(k+\alpha-j+1)\Gamma(k+\alpha+j+1)}$$

Proof. By use of the properties of Riemann-Liouville integral, the result is shown in [14]. However, a special attention should be paid the first row [16], $j=1,\cdots,m$.

$$\theta_{0,0,0}=\frac{L^\alpha\Gamma(\alpha+0.5)}{\sqrt{\pi}\Gamma(\alpha+1)^2}$$

$$\theta_{0,j,0}=\sum_{l=0}^{j}\frac{(-1)^{j-l}jL^\alpha2^{2l+1}(j+l-1)!\Gamma(l+\alpha+0.5)}{\sqrt{\pi}\Gamma(\alpha+1)(2l)!(j-l)!\Gamma(l+\alpha+1)},$$

Now, the block operational matrix of Riemann-Liouville integral [16] can be extended to fit the quasi state analysis of fractional-order integral state space model.

Definition 4. Let the augmented shifted Chebyshev vector is $\Phi(t)=\left[\phi(t)^T,\cdots,\phi(t)^T\right]^T\in\mathbb{R}^{(m+1)E}$, we have $I^n\Phi(t)=I\Phi(t)$. The block operational matrix of Riemann-Liouville integral with an order n is defined by

$$I=diag\left[I_{n_1},\cdots,I_{n_E}\right]$$

Any square-integrable vector function $F(t)\in\mathbb{R}^E$ on $[0,L]$ can be expanded in block operational matrix $F(t)=\Gamma\Phi(t)$, where $\Gamma=diag[C_1^T,\cdots,C_E^T]$. Its Riemann-Liouville integral takes $I^nF(t)=\Gamma I\Phi(t)$.

3.2 Quasi State Analysis via Block Operational Matrix

By use of the block operational matrix, the quasi state analysis of the fractional-order integral state space model is transferred into a system of linear algebraic equations.

Theorem 2. Assume the known fractional-order integral state space model (4), and

$$W(t)\simeq\Gamma\Phi(t),u(t)\simeq U^T\phi(t),U^T=[u_0,\cdots,u_m]$$

Choose $q=m+1$, the quasi state analysis problem can be transferred into a system of $E(m+1)$ linear algebraic equations.

$$\left(\Gamma-AFI-\Omega-AO\right)\Phi(t_k)=0,k=1,\cdots,q$$

where the suitable collocation points are chosen t_k,
$\Omega = diag\left[b_1U^T, \cdots, b_EU^T\right], O = diag\left[O_1^T, \cdots, O_E^T\right]$. And the quasi states can be expressed by $Z(t) = \Gamma I\Phi(t) + W(0)$

Proof. It is worth to mention the known initial conditions.

$$\bar{z}_i(0)\rho_i \simeq O_i^T\phi(t), i = 1, \cdots, E$$

$$W(0) \simeq O\Phi(t)$$

Then, substituting all the block operational matrix expansions into (4), this proof can be easily obtained.

3. NUMERICAL EXPERIMENTS

The unit step responses of two fractional-order systems are tested using fractional-order integral state space model. Block operational matrix is applied to them successfully.

In simulations, the exact solutions are compared under zero initial conditions. The time domain is set to $L = 5$. The length of Chebyshev polynomials is chosen by $m = 6, 8, 10, 12$. The linear algebraic equations are solved by *lsqlin* in Matlab 2013a. The comparison with exact quasi states is shown in Figs.1 and 2. Table 1 gives the Euclidean norm of the 50 errors.

Example 1. Consider a fractional transfer function

$$H_1(s) = \frac{s^{0.5} + 1}{s + 1}$$

According to Theorem 1, a fractional-order state space model can be realized $E = 2$.

$$n = [1, 0.5]^T, A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = B^T$$

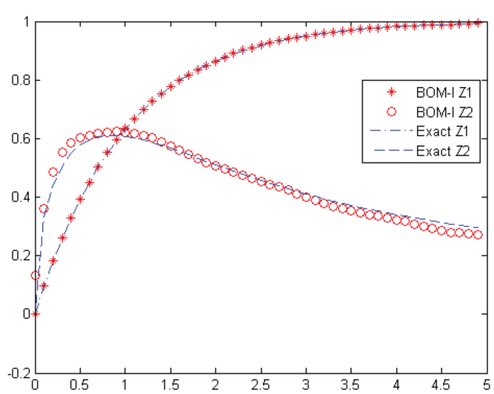

Fig. 1: The Quasi State Response with m=10

The corresponding fractional-order integral state space model is written by

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = A\begin{bmatrix} I^1w_1(t) \\ I^{0.5}w_2(t) \end{bmatrix} + Bu(t) + W(0)$$

$$y(t) = C\begin{bmatrix} I^1w_1(t) \\ I^{0.5}w_2(t) \end{bmatrix} + CW(0)$$

Choose $q = m + 1$, the collocation points are set to $t_k = Lk / q$. A system of linear algebraic equations obtained from Theorem 2 can be expanded as follows.

$$C_1^T\phi(t_k) + C_1^TI_1\phi(t_k) = U^T\phi(t_k)$$

$$C_1^TI_1\phi(t_k) + C_2^T\phi(t_k) = U^T\phi(t_k), k = 1, \cdots, q$$

The quasi states are $Z(t) = \begin{bmatrix} C_1^TI_1\phi(t) \\ C_2^TI_{0.5}\phi(t) \end{bmatrix}$.

The exact quasi states are deduced easily.

$$z_1(t) = 1 - e^{-t}$$

$$z_2(t) = \sum_{k=0}^{\infty}\frac{(-1)^k t^{k+0.5}}{\sqrt{\pi}k!}B(0.5, k+1)$$

where $B(,)$ is Beta function.

Table 1: Errors of the Quasi State Analysis with Various Choices of m

Examples	Example 1		Example 2	
	$z_1(t)$	$z_2(t)$	$z_1(t)$	$z_2(t)$
m=6	0.00611	0.23272	0.14538	0.17176
m=8	0.00449	0.20050	0.14570	0.13886
m=10	0.00660	0.18516	0.14541	0.12095
m=12	0.13841	0.30135	0.14837	0.12253

Example 2. Consider a fractional transfer function

$$H_1(s) = \frac{1}{s + s^{0.5} + 1}$$

According to Corollary 1, a fractional-order state space model can be realized $E = 2$.

$$n = [0.5, 0.5]^T, A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T$$

The corresponding fractional-order integral state space model is written by

$$\begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = A\begin{bmatrix} I^{0.5}w_1(t) \\ I^{0.5}w_2(t) \end{bmatrix} + Bu(t) + W(0)$$

$$y(t) = C\begin{bmatrix} I^{0.5}w_1(t) \\ I^{0.5}w_2(t) \end{bmatrix} + CW(0)$$

Choose $q = m + 1$, the collocation points are set to $t_k = Lk / q$. A system of linear algebraic equations obtained from Theorem 2 can be expanded as follows.

$$C_1^T\phi(t_k) - C_2^TI_{0.5}\phi(t_k) = 0, k = 1, \cdots, q$$

$$C_1^TI_{0.5}\phi(t_k) + C_2^T\phi(t_k) + C_2^TI_{0.5}\phi(t_k) = U^T\phi(t_k)$$

The quasi states are $Z(t)=\begin{bmatrix} C_1^T I_{0.5}\phi(t) \\ C_2^T I_{0.5}\phi(t) \end{bmatrix}$.

The exact quasi states are deduced easily.

$$Z(t)=\sum_{k=0}^{\infty}\frac{A^k B t^{(k+1)/2}}{\Gamma(k+1.5)}$$

From Figs. 1 and 2, an efficient approximation of the quasi states is exhibited. The errors may come from two aspects. Firstly, as there no closed solution for most fractional-order systems, the truncation of infinite series is always taken for a suitable performance. The truncation length is set to 100 in our simulations. Another error source is the block operational matrix expansion. Actually Chebyshev polynomials have been proved to approximate square-integrable functions for arbitrary precision.

In Table 1, a median length of Chebyshev polynomials can take a good approximation of the quasi states. It may depend on the complexity of the fractional-order integral state model and the initial conditions.

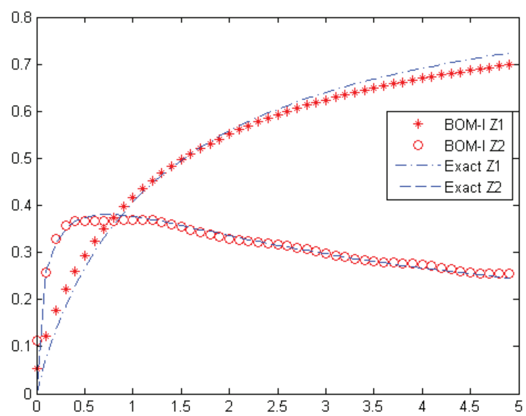

Fig. 2: The Quasi State Response with m=10

4. Conclusion

A generalized fractional-order integral state space realization for a class of any proper fractional transfer functions is given for single-input and single-out fractional-order system. Although the proposed block operational matrix is well suitable for the quasi state analysis of such fractional-order systems, it should be mentioned that the solution existence is to be proved. Besides, a large length of Chebyshev polynomials may deteriorate the approximation performance, a further improvement should be considered, such as least square optimization.

Also, other operational matrices and their block forms can be applied in fractional-order systems analysis, such as fractional-order system identification, optimal control, and sensitivity analysis. Last but not least, several misinterpretations of fractional-order state space should be noted in [15] for further improvements.

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